The 'Triptych Fractal' – A New Feature of the Logistic Map*

R. Richter[1], C. A. Kruelle[1], A. Kittel[2], and O. E. Rössler[3]

[1] Physikalisches Institut, Lehrstuhl Experimentalphysik II, Universität Tübingen, D-72076 Tübingen, FRG
[2] Physikalisches Institut, Universität Bayreuth, D-95440 Bayreuth, FRG
[3] Division of Theoretical Chemistry, University of Tubingen, 72076 Tubingen, FRG

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The recently introduced path length method for nondissipative systems is generalized to dissipative systems, specifically, the logistic map. A new prototype fractal is found to underlie the resulting nowhere-differentiable structure ("triptych principle").

1. Introduction

The logistic map is the drosophila melanogaster of chaos research. Recently, we introduced the mirror cabinet method to describe reflections in the Bunimovich stadium [1]. This method can be generalized to arbitrary dissipative systems including the logistic map.

Specifically, the total path length of a trajectory as a function of initial condition and iteration number can be focused on. Figure 1 shows the principle: The original value, labeled 0, the first increment, labeled 1, the second increment, 2, and the absolute value of the third, 3, and so on, are all well defined for every initial point, x_0. They can be stacked on top of each other along the dotted line.

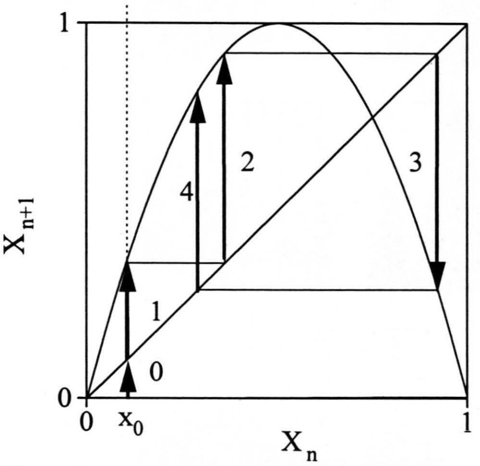

Fig. 1. Demonstration of total path length, $0+1+2+3+4+\ldots$ (absolute values), for a given initial point as a function of initial position.

2. Numerical Results

Figure 2 shows the first 8 levels of the stack ("black forest"). The underlying generating principle can be glimpsed from Figure 3. One sees how the increments labeled 0, 1, …, 4 in Fig. 1 vary lawfully in their lengths as a function of the initial condition.

One also sees that from the second iterate on, only symmetric increments contribute to the evolving fractal. Starting with the third entry, each subsequent picture is a "collage" of two identically distorted, mutually mirror-reflected, versions of the previous picture.

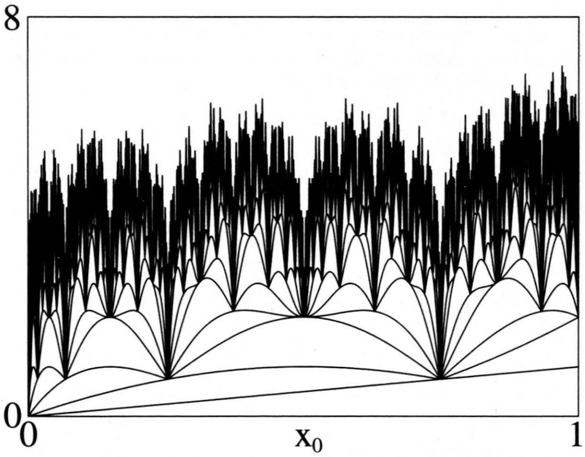

Fig. 2. The path length method of Fig. 1, applied to 10,000 adjacent initial points of the logistic map, $x_{n+1} = r\, x_n (1 - x_n)$, $r = 4$, with neighboring points of the levels graphically connected.

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Reprint requests to R. Richter, Physikalisches Institut, Experimentalphysik II, Universität Tübingen, D-72076 Tübingen.

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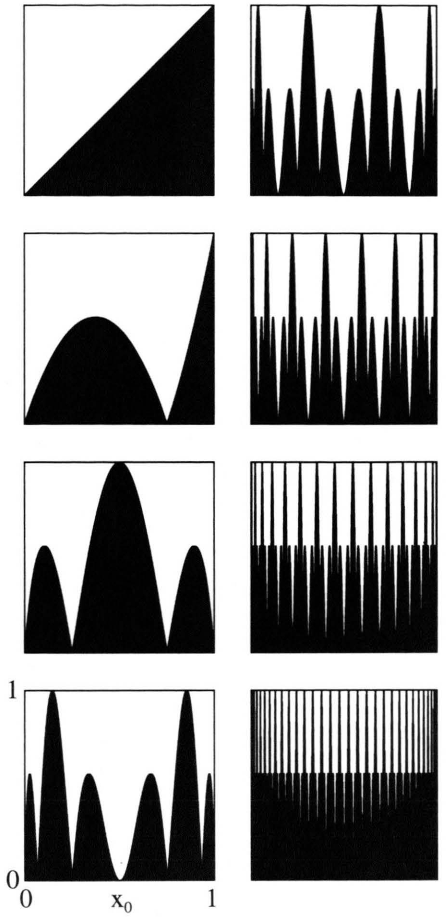

Fig. 3. The length of subsequent increment functions as a function of iteration step number: The zeroth, first, ..., up to the seventh contribution are displayed in vertical succession, starting with the upper left picture.

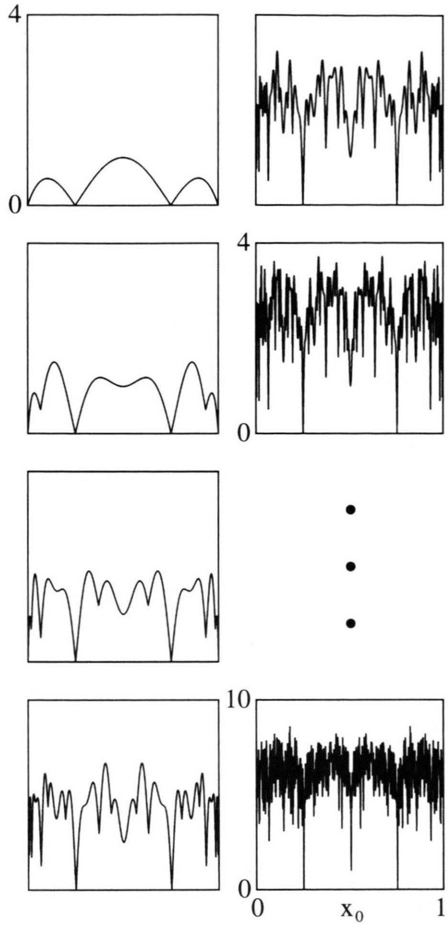

Fig. 4. The subsequent symmetric approximations to the final fractal, cf. text.

The sums of all symmetric contributions (that is, starting with the third entry of Fig. 3, omitting the first two asymmetric components) are shown in Figure 4. The final entry, lower-right corner, shows the still numerically reliable 15th iterate; note also the change of scale in the ordinate.

Quantitatively, one finds numerically that the integral under each contribution (black area) of Fig. 3 converges to a constant limiting value, 0.41349... At the same time, the total surface length of each entry doubles exactly in the limit from one step to the next. This fact implies that the area under the developing approximation to the limiting fractal, as depicted in Fig. 4, increases linearly, while its total length doubles with each step.

3. The Triptych Principle

A phenomenological explanation of the complexity of the new fractal is finally offered in Figure 5. One sees that the whole content of the upper picture (third entry in Fig. 4) is repeated, with minor distortion, in the central piece of the second picture (fourth entry of Fig. 4), and so forth.

The "central panel" of the triptych is flanked by two symmetric outer portions ("side panels"). They turn out to be nothing but distorted reflections of the two halves of the central piece. This principle is retained throughout all subsequent iterations.

The triptych principle implies that the number of peaks doubles with every step.

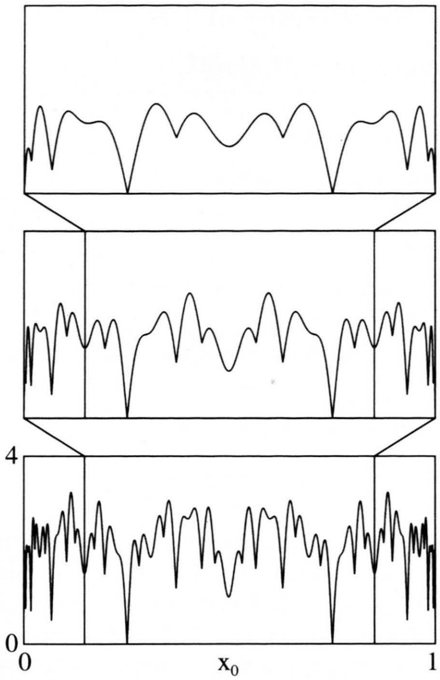

Fig. 5. The triptych composition principle in the transition between the third, fourth, and fifth picture of Figure 4.

4. Discussion

As is well known, it is possible to generate nowhere-differentiable fractal boundaries by chaotically forcing a bistable system [2, 3]. A phenomenological explanation has been proposed by Peinke et al. [4] in terms of a Weierstrass function that is formed by "layers", each containing twice as many "arcs" as it predecessor. The global analysis is still not completely worked out.

Here, surprisingly, a similar situation is found, without any second variable involved, as an implication of a particular, new way of looking at the logistic map itself. The point of departure was an idea underlying a recent numerical analysis of the Bunimovich stadium [1]. There, it turned out empirically that – if the stadium is treated as a mirror cabinet seen from inside – the total path length of the ray, starting in the center of the stadium, depends sensitively on the initial starting angle. Plotting the sequence of path lengths as a function of the angle around the central initial point, one finds a sea-urchin shape. When projecting to cartesian coordinates, a developing self-similar fractal is found [1, 5].

The same idea appears to be applicable to the analysis of dissipative systems, starting with the logistic map. The first free-of-charge results have been presented above. In addition to that, we found that the complexity of the evolving structure depends sensitively on the value of the control parameter r of the logistic map.

Of course, some difficult questions lurk in the background. What is the dimension of the limiting fractal? What is its relationship to the "chaotic-forcing" fractals described in the literature? Is there a relationship to the profile of growing interfaces in nature [6]?

Finally, it appears worth noting that this method, unlike many other methods of dealing with chaotic systems, allows an overview of the dynamics for all possible starting conditions. It therefore might be empirically applicable to time series analysis and its reconstructed attractors. Many different path lengths can be extracted from the same experimental data points by the introduction of appropriate shifts on the reconstructed attractor.

To conclude, a new "visualization approach" to dissipative chaos has been proposed. The relationship to global analysis on the one hand applications on the other remains to be explored.

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[1] C. A. Kruelle, A. Kittel, and R. P. Huebener, Z. Naturforsch. **48a,** 1039 (1993).
[2] J. Kaplan, J. Mallet-Paret, and J. A. Yorke, Ergod. Th. & Dynam. Sys. **4,** 261 (1984).
[3] O. E. Rössler, J. L. Hudson, and M. Klein, J. Phys. Chem. **93,** 2858 (1989).
[4] J. Peinke, M. Klein, A. Kittel, G. Baier, J. Parisi, R. Stoop, J. L. Hudson, and O. E. Rössler, Europhys. Lett. **14,** 615 (1991)
[5] C. A. Kruelle, in preparation.
[6] J. S. Langer, Rev. Mod. Phys. **52,** 1 (1980), and references therein; in: Dynamics of Fractal Surfaces (F. Family and T. Vicsek, eds.), World Scientific, Singapore 1991, and references therein.